École Nomale Supérieure
International Selection 2016
Major Mathematics

## Exercise 1

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence equal to one.

1. Suppose that the series $\sum_{n=0}^{\infty} a_n$ is convergent. Show that the convergence of $\sum_{n=0}^{\infty} a_n x^n$ is uniform on the segment $[0, 1]$. Deduce that $\sum_{n=0}^{\infty} a_n = \lim_{x\to 1^-} f(x)$.

2. Show by an example that in general the existence of the limit $\lim_{x\to 1^-} f(x)$ does not imply that the series $\sum_{n=0}^{\infty} a_n$ is convergent.

Suppose from now on that the limit $\lim_{x\to 1^-} f(x)$ exists and is equal to $A \in \mathbb{R}$.

Prove that the series $\sum_{n=0}^{\infty} a_n$ is convergent and $\sum_{n=0}^{\infty} a_n = A$ in each of the following three different situations:

3. if $a_n \geq 0$ for all $n$ ;

4. if $\lim_{n\to\infty} n a_n = 0$ ;

5. if $\sum_{n=0}^{\infty} n|a_n|^2$ is convergent.

## Exercise 2

For any finite set $X$, we denote by $\left|X\right|$ the number of elements of $X$.

Let $G$ and $F$ be two finite sets, and let $g \mapsto M(g)$ be a map from $G$ to the power set of $F$ (that is, to the set of all subsets of $F$).

1. Assume that there exists an injective map $i : G \to F$ satisfying $i(g) \in M(g)$ for all $g \in G$. Prove that, for any subset $H$ of $G$, one has

$$\left| \cup_{g\in H} M(g) \right| \;\geq\; \left|H\right|. \qquad (*)$$

2. Assume that Condition (*) is satisfied for any subset $H$ of $G$. Prove that there exists an injective map $i : G \to F$ satisfying $i(g) \in M(g)$ for all $g \in G$.

   Hint: one can proceed by induction considering two cases depending on whether there exists a proper non-empty subset $H$ of $G$ such that the equality holds in (*).

3. The deck of 52 cards is divided in 13 packs of four cards each. Prove that these 13 packs can be ordered in such a way that the first pack contains an ace, the second pack contains a card 2, the third pack contains a card 3 and so on (the last pack containing a king).
4. Let $A$ be a finite group, and let $B \subset A$ be a subgroup. Prove that there exists a collection $\mathcal{C}$ of elements of $A$ such that $\mathcal{C}$ is a system of representatives of the left cosets of $B$ in $A$ and at the same time $\mathcal{C}$ is a system of representatives of the right cosets of $B$ in $A$.
5. Let $m \le n$ be positive integer numbers. A *latin rectangle* of size $m \times n$ is a matrix having $m$ lines and $n$ columns such that there are $n$ different numbers among the coefficients of this matrix, each of these numbers appearing exactly once in each line and at most once in each column. Prove that each latin rectangle of size $m \times n$ can be extended to a latin square of size $n \times n$.

## Exercise 3

Let $n$ be a positive integer. Let $\mathcal{B}_n$ be the set of real $n \times n$ matrices with entries $\pm 1$. If $\mathcal{P}$ is a property satisfied by certain elements of $\mathcal{B}_n$, we call the quantity

$$\mathbb{P}(\mathcal{P}) := \frac{\left|\{M \in \mathcal{B}_n \mid M \text{ satisfies } \mathcal{P}\}\right|}{2^{n^2}}$$

the probability of $\mathcal{P}$.

1. Show that for every element $M$ of $\mathcal{B}_n$, we have
$$|\det(M)| \le n^{n/2}.$$

2. Show that
$$\sum_{M \in \mathcal{B}_n} \det(M)^2 = 2^{n^2}\, n!\,.$$

In the sequel, we will prove that $\mathbb{P}(\det(M) = 0)$ tends to 0 as $n$ tends to infinity. We denote by $M_i$, for $i = 1, \ldots, n$, the columns of $M$, and by $V_i$ the linear subspace of $\mathbb{R}^n$ generated by $M_1, \ldots, M_{i-1}$. By convention, we have $V_1 = \{0\}$.

3. Show that
$$\mathbb{P}(\det(M) = 0) \le \sum_{i=1}^{n} \mathbb{P}(M_i \in V_i).$$

4. Show that
$$\mathbb{P}(M_i \in V_i) \le 2^{-(n-i+1)}$$

for each $i = 1, \ldots, n$.

5. Let $k$ be a positive integer, and let $\mathcal{F}$ be a family of subsets of $\{1, \ldots, k\}$ such that for every pair $A, B$ of distinct elements of $\mathcal{F}$, neither $A$ contains $B$ nor $B$ contains $A$. Show that

$$|\mathcal{F}| \le \binom{k}{\lfloor k/2 \rfloor}.$$

6. Let $a_1, \dots, a_k$ be non-zero real numbers. Deduce from 5. that

$$\frac{1}{2^k}\left|\left\{(\varepsilon_1, \dots, \varepsilon_k) \in \{\pm 1\}^k \mid \sum_{l=1}^{k} \varepsilon_l a_l = 0\right\}\right| \leq Ck^{-1/2},$$

where $C > 0$ is a universal constant independent of $k$ and numbers $a_1, \dots, a_k$.

Hint: you can use Stirling asymptotic formula $k! \sim (2\pi k)^{1/2}(\frac{k}{e})^k$.

7. Show that $\mathbb{P}(\det(M) = 0)$ tends to 0 when $n$ tends to infinity.

Hint: when $i = n - o(1)$, estimate $\mathbb{P}(M_i \in V_i)$ by introducing a non-zero vector $(a_1, \dots, a_n)$ in the orthogonal complement to $V_i$. Then, let $k$ be the number of $l$'s such that $a_l \neq 0$, and consider two different cases $k \leq \log\log n$ and $k > \log\log n$.