Mathematics, Problem 2

This problem is devoted to the construction of certain subgroups of the group $\mathrm{SL}_2(\mathbb{R})$ of 2×2 matrices with coefficients in $\mathbb{R}$ and determinant 1, and of the group $\mathrm{PSL}_2(\mathbb{R})$ that we will define below.

We recall that if K is a group, the subgroup generated by elements $x_1, \ldots, x_k \in K$ is the smallest subgroup of K containing $x_1, \ldots, x_k$.

Let G be a group with neutral element e, and let H and H' be two subgroups of G. We say that G is the *free product* of H and H' if the following holds :

- G is generated by the elements of H and H'.
- Let n be a positive integer, and let $g_1, \ldots, g_n$ be n elements of $H \cup H'$, all different from e. Assume that for all $i \in \{1, \ldots, n-1\}$, either ($g_i \in H$ and $g_{i+1} \in H'$) or ($g_i \in H'$ and $g_{i+1} \in H$). Then

$$g_1 \ldots g_n \neq e.$$

If G is the *free product* of H and H', we will write $G = H * H'$.

1. Let G be a group, and assume that G is the free product of its subgroups H and H'.
   (a) Show that $H \cap H' = \{e\}$.
   (b) Assume that neither H nor H' is reduced to $\{e\}$. Show that G is *not* an abelian group.
   (c) Let $\widetilde{G}$ be a group. Let $f : H \to \widetilde{G}$ and $f' : H' \to \widetilde{G}$ be two group morphisms. Show that there exists a *unique* group morphism $\widetilde{f} : G \to G'$ such that $\widetilde{f}_{|H} = f$ and $\widetilde{f}_{|H'} = f'$.
2. Let a, b be two real numbers. Let

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \text{ and B} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

   For any nonnegative integer i, choose non-zero integers n_i and m_i. Define by induction

$$M_0 = \mathrm{Id},\ M_{2i+1} = M_{2i} A^{n_i} \text{ and } M_{2i+2} = M_{2i+1} B^{m_i}.$$

   If i is a nonnegative integer, let $c(2i)$ be the coefficient of index $(1,1)$ in M_{2i}, and let $c(2i+1)$ be the coefficient of index $(1,2)$ in M_{2i+1}, so that

$$M_{2i} = \begin{pmatrix} c(2i) & * \\ * & * \end{pmatrix} \text{ and } \mathrm{M_{2i+1}} = \begin{pmatrix} * & c(2i+1) \\ * & * \end{pmatrix}.$$

   Also let H be the subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by A and let H' be the subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by B. Let G be the subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by A and B.
   (a) Assume now that $a, b \geq 2$.
      i. Show that $|c(n)| \geq n + 1$ for all nonnegative n.
      ii. Deduce that $G = H * H'$.
      iii. Show that G is a discrete group : for every $g \in G$, there exists an open subset U of $\mathrm{M}_2(\mathbb{R})$ such that $U \cap G = \{g\}$.
   (b) Assume now that $a = b = 1$. Do we still have $G = H * H'$?
3. We now consider elements of $\mathrm{SL}_2(\mathbb{R})$ as functions from $\mathbb{R}^2$ to itself. We keep the notations above, the groups G, H, H' are defined in 2.
   (a) Show that there exist two disjoint nonempty subsets X and X' of $\mathbb{R}^2$ such that
      - if $h \in H \setminus \{\mathrm{Id}\}$, then $h(X) \subset X'$,
      - if $h \in H' \setminus \{\mathrm{Id}\}$, then $h(X') \subset X$.
   (b) Let n be a positive integer, let $h_0, \ldots, h_n$ be elements of $H \setminus \{\mathrm{Id}\}$ and let $h'_1, \ldots, h'_n$ be elements of $H' \setminus \{\mathrm{Id}\}$. Using the preceding question, show that

$$h_0 h'_1 h_1 \ldots h'_n h_n \neq \mathrm{Id}.$$

   (c) Without using the results of question 2, show again that $G = H * H'$.

4. We introduce a new symbol ∞, and define $\overline{\mathbb{R}}$ as the union of $\mathbb{R}$ and $\{\infty\}$. Let $\mathrm{PSL}_2(\mathbb{R})$ be the group of functions

$$f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$$

such that , $f(x) = \frac{ax+b}{cx+d}$ if $x \in \mathbb{R}$ and $cx + d \neq 0$, $f(x) = \infty$ if $cx + d = 0$, $f(\infty) = \frac{a}{c}$ if $c \neq 0$ and $f(\infty) = \infty$ if $c = 0$, where $(a, b, c, d) \in \mathbb{R}^4$ and $ad - bc = 1$. The group law is given by composition of functions.

   (a) Show that $\mathrm{PSL}_2(\mathbb{R})$ is indeed a group, and show that there exists a surjective morphism $\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$. What is its kernel ?

   (b) Prove that there exists only two elements S and T of $\mathrm{PSL}_2(\mathbb{R})$ such that

$$S(x) = \frac{-1}{x} \text{ and } T(x) = x + 1$$

   for $x \in \mathbb{R} \setminus \{0\}$.

   Let H be the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by S, and let H' be the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by T. Let G be the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by S and T.

   (c) Show that H and H' are finite cyclic groups.

   (d) Show that there exist two disjoint nonempty subsets X and X' of $\overline{\mathbb{R}}$ such that
   – if $h \in H \setminus \{\mathrm{Id}\}$, then $h(X) \subset X'$,
   – if $h \in H' \setminus \{\mathrm{Id}\}$, then $h(X') \subset X$.

   (e) Show that $G = H * H'$.

5. We call $\mathrm{PSL}_2(\mathbb{Z})$ the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ consisting of f such that $f(\mathbb{Z}) \subset \mathbb{Z}$.

   (a) Show that $G = \mathrm{PSL}_2(\mathbb{Z})$.

   (b) Let $\tilde{S}$ and $\tilde{T}$ be elements of $\mathrm{SL}_2(\mathbb{R})$ such that $\rho(\tilde{S}) = S$, $\rho(\tilde{T}) = T$ and denote by $\tilde{H}$ the subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by $\tilde{S}$ and $\tilde{H}'$ the subgroup generated by $\tilde{T}$. Also denote $\tilde{G}$ the subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by $\tilde{S}$ and $\tilde{T}$.

      i. Show that $\tilde{G} = SL_2(\mathbb{Z})$.

      ii. Prove that $\tilde{G}$ is *not* the free product $\tilde{H} * \tilde{H}'$.