# Mathematics

The problem is composed of different parts with varying difficulties.

Notation : $\mathbb{R}_+ = [0, +\infty)$. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}$ be a decreasing function such that

$$\lim_{x\to+\infty} \varphi(x) = 0.$$

The goal of this problem is to study the functions $f\colon \mathbb{R}_+ \to \mathbb{R}$ satisfying the following two properties :

$$\begin{cases} \forall x \in \mathbb{R}_+, \quad f(x+1) + f(x) = \varphi(x), \\ \lim_{x\to+\infty} f(x) = 0. \end{cases} \tag{1}$$

## Exercise 1 : existence and uniqueness of the solution of (1).

1. **Uniqueness**.
   Consider two solutions $f$ and $g$ of system (1), such that

   $$\forall x \in \mathbb{R}_+, \quad f(x+1) + f(x) = \varphi(x), \qquad g(x+1) + g(x) = \varphi(x),$$

   and

   $$\lim_{x\to+\infty} f(x) = 0, \qquad \lim_{x\to+\infty} g(x) = 0.$$

   (a) Let $x \in \mathbb{R}_+$. Prove by induction on the integer $n$ that

   $$\forall n \in \mathbb{N}, \quad f(x) = g(x) + (-1)^n\big(f(x+n) - g(x+n)\big).$$

   (b) Deduce (passing to the limit) that $f = g$.

2. **Existence**.

   (a) We admit that the series $\displaystyle\sum_{k=0}^{+\infty} (-1)^k \varphi(x+k)$ converges (explain why it converges if you know) and we set

   $$\begin{aligned} f(x) &= \varphi(x) - \varphi(x+1) + \varphi(x+2) - \varphi(x+3) + \varphi(x+4) - \varphi(x+5) + \cdots \\ &= \sum_{k=0}^{+\infty} (-1)^k \varphi(x+k). \end{aligned}$$

   Check that

   $$\forall x \in \mathbb{R}_+, \quad f(x+1) + f(x) = \varphi(x).$$

   (b) Prove that

   $$\forall x \in \mathbb{R}_+, \quad \varphi(x) - \varphi(x+1) \le f(x) \le \varphi(x).$$

   (c) Deduce that $\lim_{x\to+\infty} f(x) = 0$ and conclude.

**Exercise 2** : APPLICATION.

Let $f$ be defined by

$$f(x) = \int_0^1 \frac{t^x}{1+t}\, dt$$

(by definition $0^x = 1$).

1. Show that $f$ is well-defined for $x \in \mathbb{R}_+$.
2. Prove that
$$\forall x \in \mathbb{R}_+, \quad f(x+1) + f(x) = \frac{1}{x+1}.$$
3. Prove that
$$\forall x \in \mathbb{R}_+, \quad 0 \leq f(x) \leq \frac{1}{x+1}.$$
4. Deduce that
$$\forall x \in \mathbb{R}_+, \quad f(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{x+k+1}.$$

**Exercise 3** : SECOND EXAMPLE.

One denotes by $\mathbb{R}[X]$ the vector space of polynomials with real coefficients and, for any integer $n$ in $\mathbb{N}$, we denote by $\mathbb{R}_n[X]$ the sub-space of $\mathbb{R}[X]$ which consists of those polynomials with degree less or equal to $n$, of the form

$$P = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0,$$

with $a_k \in \mathbb{R}$ for $0 \leq k \leq n$. One denotes either by $P$ or $P(X)$ a polynomial.

1. **Study of a linear mapping**
   Set $e = \exp(1)$ and consider the mapping $\theta$ defined on $\mathbb{R}[X]$ by
$$\forall P \in \mathbb{R}[X], \quad \theta(P)(X) = \frac{1}{e} P(X+1) + P(X).$$
   (a) Prove that $\theta \colon \mathbb{R}[X] \to \mathbb{R}[X]$ is linear.
   (b) Check that, for all $P \in \mathbb{R}[X]$, one has $\deg(\theta(P)) = \deg(P)$. Then deduce that $\theta$ is injective (one-to-one).
   (c) Since $\deg(\theta(P)) = \deg(P)$, the following application is well defined : $\theta_n \colon \mathbb{R}_n[X] \to \mathbb{R}_n[X]$ with $\theta_n(P) = \theta(P)$ for $P$ in $\mathbb{R}_n[X]$. Prove that $\theta_n$ is bijective (one-to-one and onto).
   (d) Then deduce that for any $Q$ in $\mathbb{R}[X]$, there exists a unique $P$ in $\mathbb{R}[X]$ such that :
$$\frac{1}{e} P(X+1) + P(X) = Q(X).$$
   (e) Prove that, if the function $\varphi$ is of the form $x \mapsto Q(x)\exp(-x)$ where $Q$ is a polynomial function, then the solution of (1) is of the form $x \mapsto P(x)\exp(-x)$ where $P$ is such that $\theta(P) = Q$.
2. **Exemple**.
   Fix $n \in \mathbb{N}$ and denote by $P_n$ the unique polynomial $P_n$ such that :
$$\frac{1}{e} P_n(X+1) + P_n(X) = X^n,$$
   that is to say, such that $\theta(P_n) = X^n$.

(a) Let $n \in \mathbb{N}^*$.

i. Prove that the derivative $P_n'$ satisfies

$$P_n' = nP_{n-1}.$$

ii. Express the k-th derivative $P_n^{(k)}$ in terms of $P_{n-k}$ for $0 \leq k \leq n$.

(b) By using the Taylor formula prove that

$$P_n(X+1) = \sum_{k=0}^{n} \binom{n}{k} P_{n-k}(X), \quad \text{où } \binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

(c) Prove that

$$P_n(X) = \frac{e}{e+1} X^n - \frac{1}{e+1} \sum_{k=1}^{n} \binom{n}{k} P_{n-k}(X).$$

(d) Explain briefly how to compute $P_n(x)$ by an algorithm.