École Nomale Supérieure
International Selection 2017
Major Mathematics

The two exercises are independent.

Please, use separate sheets of paper for Exercise 1 and for the three parts of Exercise 2.

In each of the two exercices, a question could be skipped and its result assumed to solve a subsequent question.

Notations

- We use the notations $\mathbb{R}$ and $\mathbb{C}$ for the fields of real and complex numbers, respectively.
- For a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by $\mathbb{K}[T]$ the ring of polynomials in one variable T and by $\mathbb{K}[X, Y]$ the ring of polynomials in two variables X, Y.
- For $z \in \mathbb{C}$, we denote respectively by $\mathfrak{Re}(z)$ and $\mathfrak{Im}(z)$ the real and imaginary parts of z.
- We denote by i one of the square roots of -1 in $\mathbb{C}$.
- Given k, n two non-negative integers, we denote by $\binom{n}{k}$ the binomial coefficient:
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \ .$$
- If E is a finite set, then $\sharp E$ denotes the cardinal of E.

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Exercise 1

A *word* is a finite (possibly empty) sequence of letters. The empty sequence is called the trivial word and is denoted by e. Let W be the set of words formed only by letters a and b. For two words w_1 and w_2 in W, the *concatenation* of w_1 and w_2, denoted by $w_1 * w_2$, is the word obtained by writing successively w_1 and w_2. For example, for $w_1 = ab$ and $w_2 = bab$, we have $w_1 * w_2 = abbab$. Note that e is a neutral element for the operation $*$, namely, $e * w = w * e = w$ for any $w \in W$.

Let α and β be two positive real numbers. We define the *length* of a word $w \in W$ by
$$|w|_{\alpha,\beta} := p\alpha + q\beta,$$
where p (resp. q) is the number of occurrences of the letter a (resp. b) in w. For instance, if $w = abbab$, then
$$|w|_{\alpha,\beta} = 2\alpha + 3\beta \ .$$

By definition, the trivial word has length 0.

For all $t > 0$, define

$$N_{\alpha,\beta}(t) := \sharp\{w \in W \mid |w|_{\alpha,\beta} \leq t\} \ .$$

The goal of this problem is to estimate the growth of $N_{\alpha,\beta}(t)$.

(1) Compute $N_{\alpha,\alpha}(t)$.

(2) Prove that for all $t, s > 0$, one has

$$N_{\alpha,\beta}(t+s) \leq N_{\alpha,\beta}(t) N_{\alpha,\beta}(s+m)$$

where $m = \max(\alpha, \beta)$. Deduce that

$$\frac{1}{t} \log\left(N_{\alpha,\beta}(t)\right)$$

converges to some $\delta \geq 0$ when t tends to $+\infty$.

(3) Let z be a complex number. Define

$$S(z) := \sum_{w \in W} e^{-z|w|_{\alpha,\beta}} \ .$$

Show that the series $S(z)$ is absolutely convergent when $\mathfrak{Re}(z) > \delta$ and divergent when $\mathfrak{Re}(z) < \delta$.

(4) Show that for $\mathfrak{Re}(z) > \delta$, one has

$$S(z) = \frac{1}{1 - e^{-\alpha z} - e^{-\beta z}} \ .$$

Deduce that δ satisfies

$$e^{-\delta\alpha} + e^{-\delta\beta} = 1 \ .$$

(5) Show that

$$\delta \geq \frac{2\log(2)}{\alpha + \beta} \ ,$$

with equality if and only if $\alpha = \beta$.

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Exercise 2

Part I.

A polynomial $f \in \mathbb{R}[T]$ is called *hyperbolic* if all the roots of f are real. A polynomial f in $\mathbb{C}[T]$ is called *stable* if $f(z) \neq 0$ provided that $\mathfrak{Im}(z) > 0$.

Let f be a polynomial in $\mathbb{R}[T]$ of positive degree d.

(1) Prove that f is hyperbolic if and only if it is stable.

(2) Prove that if f is hyperbolic of degree $d \geq 1$ then

(2.1) the polynomial $T^d f(\frac{1}{T})$ is hyperbolic, and

(2.2) the derivative f' of f is hyperbolic.

(3) Suppose $f = \sum_{i=0}^{d} a_i T^i$ is hyperbolic. Prove the inequlaity

$$\frac{a_{k-1}}{\binom{d}{k-1}} \cdot \frac{a_{k+1}}{\binom{d}{k+1}} \leq \left(\frac{a_k}{\binom{d}{k}} \right)^2,$$

for any $1 \leq k \leq d-1$.

Part II.

Two hyperbolic polynomials $f, g \in \mathbb{R}[T]$ are said to be *interlacing* if

(i) $|\deg(f) - \deg(g)| \leq 1$ and

(ii) there exists an ordering $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \alpha_i \leq \beta_i \ldots$, where $\alpha_1, \alpha_2, \ldots$ are the roots of one of the two polynomials and $\beta_1, \beta_2, \ldots$ are the roots of the other one (counted with multiplicity).

If the inequalities in (ii) are strict, then f and g are said to be *strictly interlacing*. In other words, two hyperbolic polynomials are strictly interlacing if they have simple roots and if the open interval between any two consecutive roots of one polynomial contains a root of the other one.

For two polynomials f, g, define $W[f, g] := f'g - g'f$.

(4) Let f and g be two hyperbolic polynomials. Show that f and g are interlacing if and only if there exist hyperbolic polynomials h, f_1 and g_1 such that $f = hf_1$, $g = hg_1$ and f_1 and g_1 are strictly interlacing.

(5) Let f and $g \in \mathbb{R}[T]$ be two strictly interlacing hyperbolic polynomials. We will show below in (5.4) that either $f + ig$ or $f - ig$ is stable.

(5.1) Show that for all $(\alpha, \beta) \in \mathbb{R}^2 \backslash \{(0,0)\}$ the polynomial $\alpha f + \beta g$ is hyperbolic and has only simple roots. Deduce that $W(f, g)$ does not vanish on the real line.

(5.2) Assume by contradiction that there exist z_1 and z_2 with $\mathfrak{Im}(z_1), \mathfrak{Im}(z_2) > 0$ such that $f(z_1) + ig(z_1) = f(z_2) - ig(z_2) = 0$. Show that there exists z with $\mathfrak{Im}(z) > 0$ such that

$$2i(f(\bar{z})g(z) - f(z)g(\bar{z})) = 0 \ .$$

(5.3) Show that there exists $(\alpha, \beta) \in \mathbb{R}^2 \backslash \{(0,0)\}$ such that

$$\alpha f(z) + \beta g(z) = 0 \ .$$

(5.4) Conclude.

(6) Conversely, let f and g be two real polynomials such that $p = f + ig$ is stable.

(6.1) Show that there exist $h \in \mathbb{R}[T]$ and $p_1 \in \mathbb{C}[T]$ such that $p = hp_1$ and the roots of p_1 have negative imaginary part.

(6.2) From now on, we assume that the roots $\alpha_1, \ldots, \alpha_d$ of p have negative imaginary part. Prove that for all $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$, we have

$$2i(f(\bar{z})g(z) - f(z)g(\bar{z})) = |p(z)|^2 - |p(\bar{z})|^2 \geq 4|p(\bar{z})|^2 \cdot \mathfrak{Im}(z) \cdot \sum_{k=1}^{d} \frac{-\mathfrak{Im}(\alpha_k)}{|\bar{z} - \alpha_k|^2} \ .$$

(6.3) Deduce that f and g are hyperbolic and that $W(f, g) > 0$ on the real line.

(6.4) Show that f and g are strictly interlacing.

Hint: One may consider the derivative of the rational function f/g.

Part III.

A polynomial $f \in \mathbb{C}[X, Y]$ is called *stable* if $f(z_1, z_2) \neq 0$ for all $z_1, z_2 \in \mathbb{C}$ satisfying $\mathfrak{Im}(z_1) > 0$ and $\mathfrak{Im}(z_2) > 0$.

(7) Prove that $f \in \mathbb{C}[X, Y]$ is stable if and only if for any vector $v \in \mathbb{R}^2_{>0}$ and any $x \in \mathbb{R}^2$, the polynomial $f(x + Tv) \in \mathbb{C}[T]$ is stable.

(8) Let A, B and C be symmetric $d \times d$ matrices with real coefficients such that A and B are positive semi-definite. Prove that the polynomial

$$\det(XA + YB + C) \in \mathbb{R}[X, Y]$$

is either zero or stable.

(9) Let $f \in \mathbb{R}[X, Y]$ be a stable polynomial of degree d. Assume moreover that the equations $f(x, y) = 0$, $\frac{\partial f}{\partial X}(x, y) = 0$, and $\frac{\partial f}{\partial Y}(x, y) = 0$ do not have common solution in $\mathbb{R}^2$. Let Z be the set of solutions of the equation $f(x, y) = 0$ in $\mathbb{R}^2$.

(9.1) Show that $Z \subset \mathbb{R}^2$ is a disjoint union of smooth curves.

(9.2) Determine the number of connected components of Z, the topology of these components and the topology of $\mathbb{R}^2 \backslash Z$.