École Nomale Supérieure Paris
International Selection 2017
Minor Mathematics

Exercise 1

Let n be a positive integer number. Denote by E the vector space $\mathbb{R}_n[X]$ of real polynomials of degree $\leq n$ in one variable.

(1) What is the dimension of E? Find a basis of E.

(2) Consider the subset $Z \subset E$ formed by the polynomials that take integer values at all integer numbers. Is Z a vector subspace of E?

(3) Let $f \in E$ be a polynomial such that all the coefficients of f are integer. Prove that $f \in Z$.

(4) Is it true that the coefficients of any polynomial belonging to Z are necessarily integer?

(5) Find a polynomial $g \in E \setminus Z$ which takes integer values at $n+1$ pairwise distinct integer numbers.

(6) Let $h \in E$ be a polynomial such that it takes integer values at $n+1$ consecutive integer numbers k, $k+1$, …, $k+n$. Prove that $h \in Z$.

Exercise 2

Part I

Let I be an open interval of $\mathbb{R}$ containing 0, and let a be a positive real number. Let $f : I \to \mathbb{R}$ be an infinitely differentiable function, solution of the differential equation

$$f' = f^2 - a^2 \ .$$

(1) Show that if $f(t) = a$ or $-a$ for some $t \in I$, then f is constant on I.

(2) Show that if $f(0) > a$ (resp. $-a < f(0) < a$, $f(0) < -a$), then $f(t) > a$ (resp. $-a < f(t) < a$, $f(t) < -a$) for all $t \in I$.

(3) Assume that $f(0) \notin \{-a, a\}$. Compute f in terms of $x_0 = f(0)$ by integrating the function $\frac{f'}{f^2-a^2}$.

Hint: one can notice, first, that

$$\frac{1}{f^2 - a^2} = \frac{1}{2a}\left(\frac{1}{f-a} - \frac{1}{f+a}\right) \ .$$

Part II

Let I be an open interval of $\mathbb{R}$ containing 0, and let a be a positive real number. Let $g : I \to \mathbb{R}$ be a differentiable function satisfying the inequality

$$g' > g^2 - a^2 \ .$$

Let f be the solution of the equation $f' = f^2 - a^2$ with initial condition $f(0) = g(0)$ and maximal interval of definition. We want to show that $g(t) > f(t)$ for $t > 0$ and $g(t) < f(t)$ for $t < 0$.

(4) Assume by contradiction that there exists $t > 0$ such that $f(t) \geq g(t)$. Show that there exists $t_1 > 0$ such that $f(t_1) = g(t_1)$ and $f(t) < g(t)$ for all $t \in (0, t_1)$.

(5) Show that $f'(t_1) \geq g'(t_1)$.

(6) Conclude that $g(t) > f(t)$ for all $t > 0$ and $g(t) < f(t)$ for all $t < 0$.

(7) Suppose that g is defined on $\mathbb{R}$. Show that

$$|g(t)| \leq a$$

for all t. (One can start with $g(0)$.)

Exercise 3

For two integers $n, m \geq 0$, denote by $S(n, m)$ the number of surjections of a set of size n onto a set of size m.

The number of subsets of size k of a set with n elements is denoted by $\binom{n}{k}$.

(1) Calculate $S(n, n)$ and $S(n + 1, n)$.

(2) Show that

$$S(n, m) = \sum_{k=1}^{n} \binom{n}{k} S(n - k, m - 1).$$

(3) Show that

$$\sum_{k=0}^{m} \binom{m}{k} S(n, k) = m^n.$$

(4) Prove that

$$S(n, m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m - k)^n.$$