This exam consists of two independent parts. Please write your solutions to Part I and to Part II on different sheets of paper.

Part I

Problem 1: Buoyancy

A cylindrical block of wood of mass density ρ_w, radius R and height h is partially immersed in a liquid of mass density ρ_l. Throughout this question, the base of the cylinder will remain parallel to the surface of the liquid. Let z indicate the height of the cylinder that is not submerged.

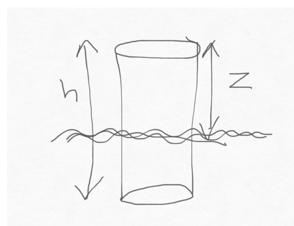


Figure 1: Partially submerged cylinder.

1. What is the equilibrium height z_{eq}?
2. If the block was initially slightly raised, so that $z(t = 0) > z_{eq}$, and then released, calculate $z(t)$, assuming no viscosity.
3. Now assume that the liquid is viscous, and that the viscous force F_v is proportional to the velocity v, i.e. $F_v = -bv$.
   (a) Write down the equation of motion.
   (b) Write down the general solution, distinguishing between three regimes for b. What type of motion occurs in each regime?
   (c) Determine the solution in each regime of b by imposing the appropriate boundary conditions.

Problem 2: An ideal gas

Consider an ideal three dimensional gas consisting of N particles of mass m at temperature T. Let the gas be confined inside a cubical box of side length L. Assume that the velocities of the particles are distributed according to the Maxwell distribution.

1. What is the normalized velocity distribution $P(v_x, v_y, v_z)dv_x dv_y dv_z$ of the particles in the gas? What is the speed distribution $P(v)dv$, $v = |\vec{v}|$?
2. What is the most probable speed of the particles (i.e. the maximum of $P(v)$)?
3. What is their average speed?
4. What is the distribution of the kinetic energy?
5. What is their most probable kinetic energy?
6. What is their average kinetic energy? Obtain this value both by calculation and by citing a physical principle.
7. What is the total energy of all particles in the box?
8. Consider instantaneously removing all particles from the gas that possess kinetic energy larger than nk_BT, with n a given real, positive number. Express your results for the following questions in terms of the error function $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx$.
   (a) How many particles remain?
   (b) What is the new total energy of the system?
   (c) After the remaining particles have returned to equilibrium, what is the new temperature of the gas?
9. Now consider the effect of the earth's gravitational field on the gas, assuming Maxwell-Boltzmann statistics. You can approximate the gravitational field as being uniform over the height L of the box. What is the average potential energy of a particle?

Problem 3: Particle in a box with delta function potential

Consider a particle of mass m moving in an infinite well with delta function potential at the origin, i.e.

$$V(x) = \begin{cases} \Lambda\delta(x) & \text{for } |x| < a\,, \\ \infty & \text{else.} \end{cases} \tag{0.1}$$

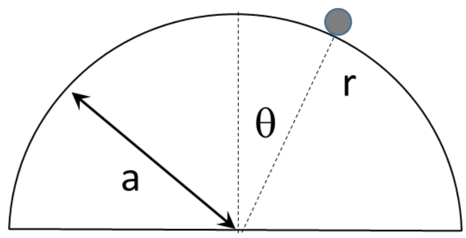
1. Write down the time-independent Schrödinger equation describing this system.
2. By studying solutions of this equation, find the value of Λ for which the ground state energy of the system vanishes.
3. The node theorem states that for one dimensional systems, the number of nodes (zeros) of the n^{th} eigen wave function (in the case considered here the zeros are counted in the region $|x| < a$) is $n - 1$. Using this input, find the energy of the first excited state of the system.
4. Sketch the wave function for the second excited state, explaining your reasoning.

Part II

1) Problem 1

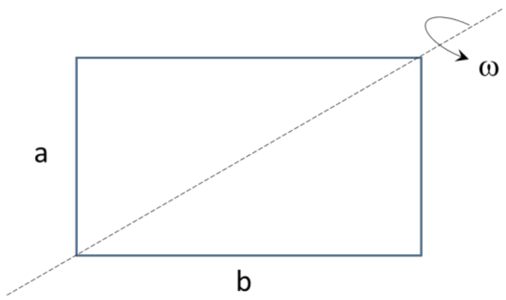
The particle (of point mass m) is sliding down from the top of the hemisphere of radius a, subject to the earth's gravity.

Find: a) normal force exerted by the hemisphere on the particle; b) angle with respect to the vertical at which the particle will leave the hemisphere.



2) Problem 2

A uniform 2D rectangular plane lamina of mass m and dimensions a and b (assume b > a) rotates with the constant angular velocity ω about a diagonal. Ignoring gravity, find: a) principal axes and moments of inertia; b) angular momentum vector in the body coordinate system; c) external torque necessary to sustain such rotation.



Problem 3

A particle (point mass m) is moving in three dimensional space, subject to the potential $U(\mathbf{r}) = kr$, where k is a constant, and $\mathbf{r} = r\mathbf{e}_r$ with $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ and $\mathbf{e}_r$ a unit vector pointing from the origin to the position $\mathbf{r}$.

1. For what energy and momentum will the orbit be a circle of radius r about the origin?
2. What is the frequency of this circular motion?
3. If the particle is slightly disturbed from this circular motion, what will be the frequency of small oscillations in radial direction?

Problem 4

A quantum particle moving in three dimensional space is subject to a potential $U(r)$ depending only on the distance from the origin (that is, we have $\mathbf{r} = r\mathbf{e}_r$ as in the preceding problem).

1. Write down the equation satisfied by the wave function $\Psi(\mathbf{r})$ of the particle.
2. Explain precisely why and how it is possible to construct the general solution to this equation from wave functions Ψ of the form $\Psi(\mathbf{r}) = \psi(r)\chi(\theta, \phi)$.
3. We recall the expression of the Laplace operator in spherical coordinates

$$\Delta = \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{1}{r^2\hbar^2}\hat{L}^2 \tag{1}$$

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \tag{2}$$

   Derive a differential equation for $\psi(r)$.

4. We now consider the specific potential

$$U(r) = \frac{A}{r^2} - \frac{B}{r} \tag{3}$$

   Write down the differential equation satisfied by ψ.

5. Give explicit expressions for the energy levels of the problem in terms of A and B. (Hint: can you use your knowledge about the energies of the hydrogen atom?)