EARTH SCIENCES
SCIENTIFIC BACKGROUND ASSESSMENT

Tornadoes and a water basin

In this exercise we will compute some of the caracteristics of the dynamics of the water in a basin. It is a – loose – approximation of some of the vortex dynamics that takes place in the Earth atmosphere and oceans, for example tornadoes.

The basin is perfectly circular, its geometry on the radial plan can be seen in Fig.1. The water at the interior of the basin has been stirred like in a cup of tea, and rigidly rotates with a constant angular speed Ω s^{-1}. We will call $\mathbf{u} = [u_r, u_\theta, u_z]$ the velocity vector and its radial, tangential and vertical components.

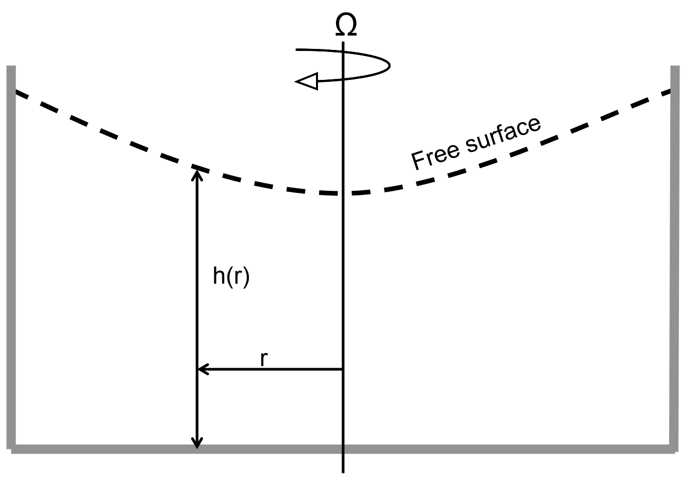

Figure 1: Geometry of the water basin

We will consider the water to be incompressible and hydrostatic (i.e. the pressure at a given location only depends on the weight of the liquid column above). This latter approximation can be written:

$$\frac{\partial p}{\partial z} = -\rho g \tag{1}$$

where p is pressure, ρ is density, and g is the gravitational acceleration. z is the vertical coordinate, defined to be zero at the bottom of the basin.

We will neglect surface tension and all friction, except the friction on the walls of the basin. In these approximations, we can write an equation for the radial balance of forces in the basin:

$$\frac{\partial u_r}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - k_r u_r + \Omega^2 r, \tag{2}$$

where k_r is a friction coefficient measured in s^{-1} that will be negligible away from the walls and the bottom of the basin.

1) Using eq.1, write an expression for the pressure at the bottom of the basin, as a function of the liquid height $h(r)$. We will call p_{atm} the atmospheric pressure.

2) At locations far from the walls of the basin, the pressure gradient force and the centrifugal force are in equilibrium. In this hypotheses, starting from eq.2, deduce an equation for the shape of the free surface of the water as a function of r.

3) The free surface equation computed at point (2) above is only valid near the center of the basin, where the friction of the walls is negligible. Qualitatively, how do you think the wall will affect the shape of the free surface? Make a schematic drawing.

4) At the bottom of the basin, the friction is not negligible. We can suppose that the layer of liquid directly in contact with the cup has negligible tangential velocity.

4.1) This has the effect of creating a radial velocity near the bottom. Explain why.

4.2) Again using eq.2, compute an expression for the radial speed as a function of radius r, as a balance between the pressure gradient force and the friction.

5) By mass conservation, the radial velocity at the bottom will create a vertical velocity along the axis of the basin. This will induce the so called secondary circulation in the radial plan of the basin. Draw it qualitatively this circulation.

The secondary circulation is resposible for the spin-down of the rotation, by subtracting energy to the vortical velocity in a much more effective way then simple friction.

6) Since the fluid is incompressible, the velocity vector is nondivergent. In our case this can in be written in the radial radial plan as:

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0. \tag{3}$$

Applying this equation, compute the vertical derivative of the vertical velocity on the axis of the basin. This estimation will be valid near the bottom.

7) In Figure 2, the red dots represent measurements of the tangential wind speed in a tornado as a function of the distance r from the axis of rotation. Ignore the black line, which is given by a numerical simulation of the same tornado.

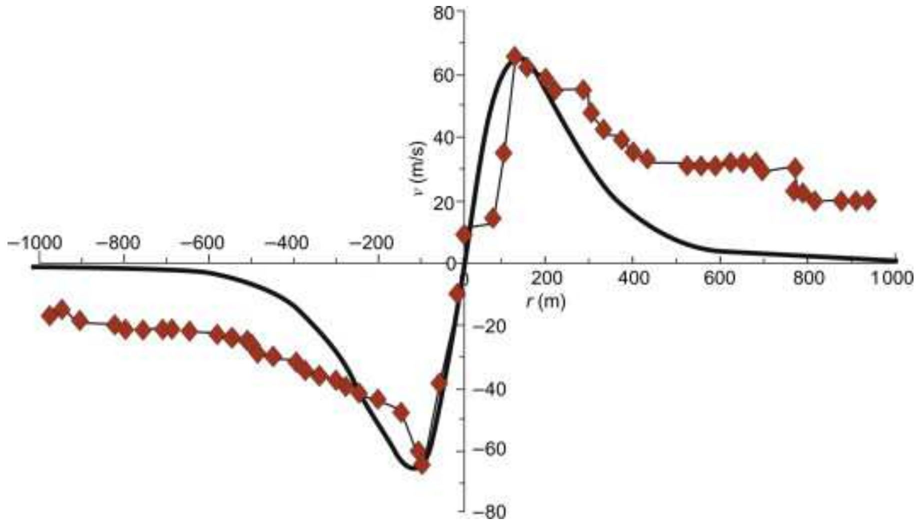

Figure 2: red dots: measurements of tangential speed in a tornado, near the surface, as a function of distance from the axis of rotation

7.1) From the figure, estimate approximatively the angular speed Ω in the proximity of the center of hte tornado.

7.2) Compute the vertical wind velocity at 2 m of elevation from the ground, at the center of the tornado. You will take a value of $k_r = 10^{-2}\ s^{-1}$ for the bottom friction coefficient.

In flat terrain, the bottom friction is low and the danger of a tornado lifting debris from the ground is higher.

figure credit: Sergey A. Arsenyev: Mathematical modeling of tornadoes and squall storms. Geoscience Frontiers Volume 2, Issue 2, April 2011, Pages 215221