You will treat the following exercises and present them both, in the order of your choice. The preparation time is 30 min; the oral examination will last an hour.

At the beginning of the oral examination, you will have a maximum of ten minutes to present your results, without the jury's intervention (you may use less than ten minutes if you wish). We encourage you not to copy all of your calculations, but rather to concentrate on the crucial points of your reasoning.

The jury will then come back to the questions that it wishes to examine in greater depth, including those that you may not have had time to address during the preparation.

We emphasize that the exercises are only a pretext for a mathematical discussion, and it is the discussion that counts. It is perfectly normal if you encounter difficulties in solving the exercises, and to arrive at the end of the exercises is not the principal objective.

* * *

Exercise 1. Let q be a quadratic form on a finite dimensional $\mathbb{R}$-vector space E. Let $s \in \mathbb{R}^*$ such that $q^{-1}(s) = A$ is non empty. Prove that A generates E, in the sense that $\mathrm{Vect}(A) = E$.

* * *

Exercise 2. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that for all integers $m, n \geq 1$ we have $a_{n+m} \leq a_n + a_m$. Set

$$\ell = \inf_{n\geq 1} \frac{a_n}{n} \in \mathbb{R} \cup \{-\infty\}.$$

(1) Show that $a_n/n \to \ell$ as $n \to \infty$.

Let $n \geq 1$ be an integer. By definition, a simple path of length n is a sequence $X_0, \dots, X_n \in \mathbb{Z}^2$ of pairwise different points of the plane with integer coordinates such that $X_0 = (0, 0)$, for every $0 \leq i \leq n-1$ the Euclidean distance between X_i and X_{i+1} is equal to 1. Denote by c_n the number of simple paths of length n.

(2) Show that $c_n^{1/n}$ converges to a value $c \in [2, 3]$ as $n \to \infty$.

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* * *

Exercise 1. Let $\mathbb{N}_{\geq 1} = \{1, 2, 3, \ldots\}$ be the set of all positive integers. The level of a field K, denoted by $\nu(K)$, and which belongs to $\mathbb{N}_{\geq 1} \cup \{\infty\}$, is defined by:

$$\nu(K) = \inf\{n \in \mathbb{N}_{\geq 1} | \exists (x_1, ..., x_n) \in K^n, x_1^2 + \cdots + x_n^2 = -1\}.$$

(1) Prove that two isomorphic fields have the same level. Is the converse true?

(2) Let K be a field with finite level. Prove that $\nu(K) = \nu(K(X))$, where $K(X)$ denotes the field of rational functions over K.

(3) Compute the level of the following fields : $\mathbf{F}_{p^n}, \mathbb{Q}(i\sqrt{2}), \mathbb{Q}(e^{2i\pi/3})$ and $\mathbb{Q}(e^{2i\pi/3}2^{1/3})$.

* * *

Exercise 2. Study the convergence of the following integrals

(1)

$$\int_0^{+\infty} \cos(x^3 - x)\mathrm{d}x$$

(2)

$$\int_0^{+\infty} \frac{\mathrm{d}x}{1 + x^p |\sin(x)|},$$

for fixed $p \in \mathbb{R}$.

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Exercise 1. For a fixed $p > 0$, find the asymptotics, as $A \to +\infty$, of the following integral:

$$\int_0^1 (1 - x^p)^A \mathrm{d}x.$$

* * *

Exercise 2. Let G be a finite group and let $n \geq 1$ be an integer which divides the order of G. Let $a_n(G)$ be the cardinal of $\{x \in G, x^n = 1\}$. Assume that G is a direct product of cyclic groups. Compute $a_n(G)$.

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* * *

Exercise 1. Let $f : [1, +\infty) \to \mathbb{R}$ be a C^2 function. For $n \geq 1$, let $f_n : [1, +\infty) \to \mathbb{R}$ be the function defined by

$$f_n(x) = \frac{n}{x} \cdot \left(f\left(x + \frac{x}{n}\right) - f(x) \right).$$

Assume that the function $x \mapsto x f''(x)$ is bounded

(1) Show that
$$\lim_{n\to\infty} \sup_{x\geq 1} |f_n(x) - f'(x)| = 0.$$

(2) Assume that $f(x)/x$ converges as $x \to +\infty$ to a limit denoted by L. Show that $f'(x) \to L$ as $x \to \infty$.

(3) Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that $g'(x)$ converges as $x \to +\infty$ to a limit denoted by L. Is it true that $g(x)/x \to L$ as $x \to \infty$? Conversely, if $g(x)/x$ converges to L as $x \to +\infty$, is it true that $g'(x) \to L$ as $x \to \infty$?

* * *

Exercise 2. Let $n \geq 1$ an integer. We denote by $\mathsf{Gl}_n(\mathbb{Z})$ the group of all $n \times n$ invertible matrices with integer coefficients equiped with the multiplication operation. Prove that $\mathsf{Gl}_n(\mathbb{Z})$ contains, up to isomorphism, only finitely many finite subgroups.

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* * *

Exercise 1. Fix $M > 1$. We consider a sequence $(A_n)_{n\geq 1}$ of matrices such that for every $n \geq 1$:

(i) A_n is a $n \times n$ matrix with integer coefficients;

(ii) The sum of the absolute values of the entries of any row of A_n is at most M.

For every $\delta > 0$, denote by $N_\delta(A_n)$ the number of eigenvalues of A_n of modulus strictly between 0 and δ.

(1) Show that if $\lambda \neq 0$ is an eigenvalue of A_n then $|\lambda| \leq M$.

(2) Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \geq 1$ we have

$$N_\delta(A_n) \leq \varepsilon n.$$

* * *

Exercise 2. Let $a \in \mathbb{R}^n$. We denote by $\mathcal{C}^\infty(\mathbb{R}^n)$ the set of infinitely differentiable functions on $\mathbb{R}^n$. Find all functions $F : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$, verifying the following three properties:

(1) $F(f + g) = F(f) + F(g)$ for any $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$,

(2) $F(\alpha f) = \alpha F(f)$ for any $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$,

(3) $F(fg) = F(f)g(a) + f(a)F(g)$.

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* * *

Exercise 1. Prove that for any complex numbers $a_1, \dots, a_n$ and positive semi-definite complex matrices $A_1, \dots, A_n$, the following inequality is satisfied:

$$\det\Big(|a_1|A_1 + \cdots + |a_n|A_n\Big) \geq \Big|\det\Big(a_1A_1 + \cdots + a_nA_n\Big)\Big|.$$

* * *

Exercise 2. Let $n \geq 2$ be an integer. We consider a sequence $(X_i)_{1\leq i\leq n}$ of independent random variables which all follow the uniform distribution on the set $\{1, 2, \dots, n\}$. We define

$$M_n = \min_{1\leq i\leq n-1} |X_{i+1} - X_i|.$$

1) Show that the sequence $(\mathbb{P}(M_n = 0))_{n\geq 2}$ converges and compute its limit.

2) Show that the sequence $(\mathbb{E}[M_n])_{n\geq 2}$ is bounded.

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* * *

Exercise 1. Prove that for any positive real numbers $a_1, \dots, a_n$, verifying $a_1 + \cdots + a_n = 1$, and positive definite complex matrices $A_1, \dots, A_n$, the following inequality is satisfied:

$$\det\Big(a_1 A_1 + \cdots + a_n A_n\Big) \geq \Big(\det A_1\Big)^{a_1} \cdots \Big(\det A_n\Big)^{a_n}.$$

* * *

Exercise 2. Fix $c > 0$. We consider a sequence $(a_n)_{n\geq 1}$ of real numbers such that $0 < a_n \leq c(a_{2n} + a_{2n+1})$ for every $n \geq 1$. For which values of c is the series $\sum_{n\geq 1} a_n$ always divergent?